Economics Letters 196 (2020) 109483

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Profit efficiency: Generalization, business accounting and the role of convexity

Rolf Färe [a], Valentin Zelenyuk [b,*]

[a] Department of Economics, Oregon State University and Department of Agricultural and Resource Economics, University of Maryland, USA
[b] School of Economics and Centre for Efficiency and Productivity Analysis (CEPA) at The University of Queensland, Australia

ARTICLE INFO

Article history:
Received 28 June 2020
Received in revised form 12 August 2020
Accepted 12 August 2020
Available online 18 August 2020

JEL classification:
D24
C43
L25

Keywords:
Profit efficiency
Nerlovian efficiency

ABSTRACT

We present new theoretical results, showing that Nerlovian profit efficiency is a special case of the recently introduced general profit efficiency, deriving a new decomposition of profit efficiency, and outlining a simple way of estimating the generalized profit efficiency measure.
© 2020 Elsevier B.V. All rights reserved.

1. Theoretical underpinnings

Profit of a business activity is considered to be the main goal of any for-profit organization. For this very reason, profit maximization criterion is also the corner stone of virtually any model in mainstream economic theory. Formally, the profit-maximization framework typically starts with a premise that the main goal is the (maximal) profit, defined as

$$\pi(\Psi_t) = \sup_{x,y} \{p(y)y - w(x)x \ : \ (x, y) \in \Psi_t\}, \tag{1.1}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N_+$ and $y = (y_1, \ldots, y_M) \in \mathbb{R}^M_+$ are column vectors of inputs and outputs, respectively, and $w(x) = (w_1, \ldots, w_N) \in \mathbb{R}^N_+$ and $p(y) = (p_1, \ldots, p_M) \in \mathbb{R}^M_+$ are row vector functions of their corresponding prices, which in general are functions of the quantities of inputs demanded and quantities of outputs supplied, and Ψ_t is the relevant technology set characterizing what is feasible to produce at time t and defined, in very general terms, as

$$\Psi_t \equiv \{(x, y) \ : \ x \text{ can produce } y\}. \tag{1.2}$$

Various assumptions are usually made on the technology set, Ψ_t, sometimes referred to as regularity conditions of production theory and typically include free disposability (or monotonicity) of inputs and outputs, no free lunch, feasibility of no activities, closedness, boundedness of the outputs set and so on (e.g., see Sickles and Zelenyuk, 2019), which we accept here.

Furthermore, when concerned with measuring relative efficiency, a common benchmark relative to which all firms can be measured or referenced to is useful. A natural and perhaps the ultimate benchmark for economic efficiency measurement is the one implied by the perfectly competitive market equilibrium under full information, which implies that the input and output prices are exogenous to a firm in its decision making on (x, y) in (1.1). I.e., the benchmark is given by

$$\pi(w, p|\Psi_t) = \sup_{x,y} \{py - wx \ : \ (x, y) \in \Psi_t\}, \tag{1.3}$$

where the row vectors $w = (w_1, \ldots, w_N) \in \mathbb{R}^N_+$ and $p = (p_1, \ldots, p_M) \in \mathbb{R}^M_+$ now do not depend on x and y. In this paper we will also focus on the static case and therefore drop the subscript t, leaving the dynamic case (e.g., those involving Bellman equations) for future research.

Many profit efficiency measures have been introduced in the literature, e.g., see Färe et al. (2019) for a review and references. A general measure of *profit efficiency*, can be stated for an observed quantity vector (x^j, y^j) and price vector (w, p), as following

$$\mathscr{E}(x^j, y^j; w, p|\Psi) = \sup_{\boldsymbol{\theta}, \boldsymbol{\lambda}, x, y} \{\psi(\boldsymbol{\theta}, \boldsymbol{\lambda}) \ : \ \psi_y(p, y^j, \boldsymbol{\theta}) - \psi_x(w, x^j, \boldsymbol{\lambda}) \le py - wx, \tag{1.4}$$

* Correspondence to: 530, Colin Clark Building (39), St Lucia, Brisbane, Qld 4072, Australia.
E-mail address: v.zelenyuk@uq.edu.au (V. Zelenyuk).

https://doi.org/10.1016/j.econlet.2020.109483
0165-1765/© 2020 Elsevier B.V. All rights reserved.

$$(x, y) \in \Psi,\ \boldsymbol{\theta} \in \Theta,\ \boldsymbol{\lambda} \in \Lambda\}$$

where $\psi(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is an objective function chosen by a researcher, to be optimized jointly over $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ and (x, y), while $\psi_x(w, x, \boldsymbol{\lambda})$ and $\psi_y(p, y, \boldsymbol{\theta})$ are the functions that define the way how the measurement of the efficiency is conveyed with regard to each element of input vector x and of output vector y, and where Θ and Λ are the set of permissible values of $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$.[1]

It is worth noting that the formulation (1.4) is slightly more general than the original that appeared in Färe et al. (2019), who were more explicit, by letting $\psi_y(p, y^j, \boldsymbol{\theta}) = \sum_{m=1}^{M} p_m\theta_m y^j_m$ and $\psi_x(w, x^j, \boldsymbol{\lambda}) = \sum_{i=1}^{N} w_i\lambda_i x^j_i$, with $\Theta = \mathbb{R}^M_+$ and $\Lambda = \mathbb{R}^N_+$. This more general formulation (1.4) is also a way to formalize what one may think of as 'producer preferences' stated via an optimization criterion that is more general than a profit function, which subsumes the latter as a special case and allows for incorporating various objectives of the firm.[2]

This general profit efficiency measure can be viewed as a dual analogue of a technical efficiency measure that satisfies Pareto–Koopmans efficiency criterion, but in addition to taking into account efficiency with respect to technology it also takes into account the quintessence of the market environment where the firm operates. That is, it also incorporates the prices that reflect the current valuations of both buyers and sellers on those markets. In this sense, this measure can be viewed as one that satisfies a superior criterion relative to the (primal) Pareto–Koopmans efficiency criterion that only considers technical efficiency.

Many measures of efficiency known in the literature can be deduced from (1.4) directly or after some transformations, as was detailed in Färe et al. (2019). This also includes the directional distance functions and the so-called Nerlovian or directional profit efficiency measure (Chambers et al., 1998), defined as

$$\Pi E(x^j, y^j, p, w|-d_x, d_y, \Psi) \equiv \frac{\pi(w, p|\Psi) - (py^j - wx^j)}{(pd_y + wd_x)}, \tag{1.5}$$

where $(-d_x, d_y)$ is a nonzero vector in $\mathbb{R}^N_- \times \mathbb{R}^M_+$ that characterizes the direction in which the distance between (x^j, y^j) and the frontier of Ψ is to be measured. Indeed, if we let $\psi(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \alpha$, such that $\boldsymbol{\theta} = \alpha d_y$ and $\boldsymbol{\lambda} = \alpha d_x$ and set $\psi_y(p, y^j, \boldsymbol{\theta}) = p(y^j + \alpha d_y)$ and $\psi_x(w, x^j, \boldsymbol{\lambda}) = w(x^j - \alpha d_x)$, then we get

$$\begin{aligned}
&\mathscr{E}(x^j, y^j; w, p|\Psi)\\
&= \sup_{\alpha}\left\{\sup_{x,y}\left\{p(y^j + \alpha d_y) - w(x^j - \alpha d_x) \le py - wx \ :\ (x, y) \in \Psi\right\}\right\}\\
&= \sup_{\alpha}\left\{\sup_{x,y}\left\{\frac{(py - wx) - (py^j - wx^j)}{pd_y + wd_x} \ :\ (x, y) \in \Psi\right\} \ge \alpha\right\}\\
&= \sup_{\alpha}\left\{\frac{\pi(w, p|\Psi) - (py^j - wx^j)}{pd_y + wd_x} \ \ge \alpha\right\}\\
&= \frac{\pi(w, p|\Psi) - (py^j - wx^j)}{pd_y + wd_x}.
\end{aligned}$$

Moreover, a particularly interesting case of (1.4) is derived by setting $\lambda_1 = \lambda_2 = \ldots = \lambda_N = 1$ and $\theta_1 = \theta_2 = \cdots = \theta_M = \theta$ and $\psi(\lambda_1, \ldots, \lambda_N; \theta_1, \ldots, \theta_M) = \theta$, while $\psi_y(p, y^j, \boldsymbol{\theta}) = \sum_{m=1}^{M} p_m\theta_m y^j_m$ and $\psi_x(w, x^j, \boldsymbol{\lambda}) = \sum_{i=1}^{N} w_i\lambda_i x^j_i$ to obtain

$$\mathscr{E}_o(x^j, y^j; w, p|\Psi) = \sup_{\theta,x,y}\{\theta \ : \ py^j\theta - wx^j \le py - wx,\ (x, y) \in \Psi\} \tag{1.6}$$

which in turn can be stated in terms of the sup-sup (or 'maximax'), as

$$\mathscr{E}_o(x^j, y^j; w, p|\Psi) = \sup_{\theta}\left\{\sup_{x,y}\left\{\frac{py - wx + wx^j}{py^j} \ :\ (x, y) \in \Psi\right\} \ge \theta\right\}. \tag{1.7}$$

While looking very complicated and 'too theoretical', Färe et al. (2019) also derive several very intuitive versions of this measure, that are composed of simple and intuitive notions often used in business analysis. Specifically, they showed that

$$\mathscr{E}_o(x^j, y^j; w, p|\Psi) = \frac{c^j}{r^j} + \frac{\pi(p, w|\Psi)}{r^j}, \tag{1.8}$$

where $c^j = wx^j$, $r^j = py^j$ are the observed total costs and total revenue of the firm at the allocation (x^j, y^j) that face prices (w, p).[3]

That is, intuitively, (1.8) says that the profit efficiency measure defined in (1.6) or (1.7) can be decomposed into two key performance indicators used in business analysis: (i) the realized cost-benefit ratio and the best possible profit margin for the firm with allocation (x^j, y^j) that faces prices (w, p). Note that the first component is the reciprocal of the "return to the dollar" measure of performance advocated by Georgescu-Roegen (1951).[4]

Another useful decomposition, which appears to be new, can be obtained by further decomposition of the observed cost benefit ratio, to get

$$\begin{aligned}
\mathscr{E}_o(x^j, y^j; w, p|\Psi) = \ &\frac{\pi(p, w|\Psi)}{r^j}\\
&+ CE(x^j, y^j, w|\Psi) \times RE(x^j, y^j, p|\Psi)\\
&\times CBR^*(x^j, y^j, w, p|\Psi),
\end{aligned} \tag{1.9}$$

where $CE(x^j, y^j, w|\Psi)$ is the cost efficiency measure for the allocation (x^j, y^j), technology Ψ and input prices w, defined as

$$CE(x^j, y^j, w|\Psi) = \frac{wx^j}{C(y^j, w|\Psi)}, \tag{1.10}$$

where $C(y, w|\Psi)$ is the classical cost function defined as

$$C(y, w|\Psi) = \min_x\{wx \ :\ (x, y) \in \Psi\}, \tag{1.11}$$

while $RE(x^j, y^j, p|\Psi)$ is the revenue efficiency measure for allocation (x^j, y^j), technology Ψ and output prices p, defined as

$$RE(x^j, y^j, p|\Psi) = \frac{R(x^j, p|\Psi)}{py^j}, \tag{1.12}$$

and $R(x, p|\Psi)$ is the classical revenue function defined as

$$R(x, p|\Psi) = \max_y\{py \ :\ (x, y) \in \Psi\}, \tag{1.13}$$

and, finally, $CBR^*(x^j, y^j, w, p|\Psi)$ is the cost benefit ratio involving the optimal cost (from cost-minimization perspective (1.11)) and

---

[1] Some constraints may be needed on Ψ (in addition to standard regularity conditions), e.g., to regularize the non-decreasing returns to scale cases, where profit maximization may diverge to ∞ or degenerate to 0. See Färe et al. (2019) for more details.

[2] E.g., see discussion in Rubinstein (2006, p. 83–84) regarding the importance of incorporating other criteria than just profit. We thank the anonymous referee for this insight.

[3] It should be clear that this measure can also be deduced from Nerlovian profit efficiency measure by setting $d_y = y^j$ and $d_x = 0$ and adding 1 (see Färe et al., 2019 for further details).

[4] Färe et al. (2019) also show that this profit efficiency measure can be further decomposed into three sources: (i) revenue efficiency, (ii) Farrell technical efficiency (output oriented here) and (iii) a new allocative efficiency measure measuring the gap between profit maximization and revenue maximization.

optimal revenue (from revenue-maximization perspective (1.13)). Various other decompositions can be derived in a similar manner, for example for an input oriented analogue or for a generalized hyperbolic analogue or for a specific direction and we leave these to the readers. E.g., for a related extensive discussion that links efficiency and productivity analysis with business and accounting measures, see Grifell-Tatjé and Lovell (2015) and references therein.

In practice, a researcher typically does not observe Ψ and so cannot obtain the true value of $\pi(w, p|\Psi)$ and thus of $\mathscr{E}_o(x^j, y^j; w, p|\Psi)$, yet the analytical developments in the next section allow an easy computation.

2. DEA and FDH estimation under the same prices

Introduced by Farrell (1957), generalized and popularized by Charnes et al. (1978) and elaborated further in a rich interdisciplinary literature, DEA estimator found many applications in both academic and industry research. Statistical properties (consistency and limiting distributions) of DEA are also well documented (Simar and Wilson, 2015). While there is a myriad of variations of DEA, here we will focus on the most popular version, which assumes variable returns to scale (VRS).

To facilitate further discussion, let $x^k = (x_1^k, \ldots, x_N^k) \in \mathbb{R}_+^N$ and $y^k = (y_1^k, \ldots, y_M^k) \in \mathbb{R}_+^M$ be observations on inputs and outputs for a decision making unit $k \in \{1, \ldots, n\}$, then the DEA formulation for *the maximal profit function* can given by

$$\widehat{\pi}(w, p|\text{DEA-VRS}) \equiv \max_{\substack{x_1,\ldots,x_N,\ y_1,\ldots,y_M \\ z^1,\ldots,z^n}} \sum_{m=1}^{M} p_m y_m - \sum_{l=1}^{N} w_l x_l, \tag{2.1}$$
$$\text{s.t.} \quad \sum_{k=1}^{n} z^k y_m^k \geq y_m,\ m = 1, \ldots, M,$$
$$\sum_{k=1}^{n} z^k x_l^k \leq x_l,\ l = 1, \ldots, N,$$
$$\sum_{k=1}^{n} z^k = 1,$$
$$z^k \geq 0,\ k = 1, \ldots, n.$$

While there are different approaches to show what this optimization problem reduces to, here we will use the strategy employed by Färe and Grosskopf (1985) for the cost efficiency and its revenue analogue in Zelenyuk (2020). To do so, first note that because the optimization is done over all x and all y, for a continuous function over a closed set, with strictly positive prices for inputs and outputs, there exists an optimum where all the inequalities turn to equalities (i.e., no slacks). Thus, one can multiply each mth output equality constraint by the corresponding output price p_m, $(m = 1, \ldots, M)$ and sum these inequality constraints over m; similarly, one can multiply each ith input equality constraint (2.1) by the corresponding input price w_l, $(l = 1, \ldots, N)$, while keeping the other constraints the same, to get the equivalent to the following optimization problem

$$\widehat{\pi}(w, p|\text{DEA-VRS}) = \max_{z^1,\ldots,z^n} \sum_{k=1}^{n} z^k r^k - \sum_{k=1}^{n} z^k c^k,$$
$$\text{s.t.} \quad \sum_{k=1}^{n} z^k = 1,$$
$$z^k \geq 0,\ k = 1, \ldots, n,$$

where $r^k = py^k$ and $c^k = wx^k$.

Finally, let $\pi^k := r^k - c^k$ and note that we have $\sum_{k=1}^{n} z^k r^k - \sum_{k=1}^{n} z^k c^k = \sum_{k=1}^{n} z^k \pi^k$ and so the last optimization problem simplifies to:

$$\widehat{\pi}(w, p|\text{DEA-VRS}) = \max\{\pi^1, \ldots, \pi^n\}. \tag{2.2}$$

That is, one can compute $\widehat{\pi}(w, p)$ without information on (w, p) and even without information on (x^j, y^j) when all firms are assumed to face the same prices for inputs and outputs (e.g., equilibrium prices or average prices). In particular, one can simply use the aggregated information about costs and revenue, (c^j, r^j), for each j to compute the observed profit for each j, rank the computed profits across all j and select the highest value, which will be $\widehat{\pi}(w, p)$ for the given sample. This result is especially useful for cases when the dimensions of x and y are very large, e.g., including 'big data' cases, because the DEA estimator is known as not immune from the so-called 'curse of dimensionality' problem.

Also note that the same result holds for the case of DEA with decreasing returns to scale, i.e., when $\sum_{k=1}^{n} z^k = 1$ is replaced with $\sum_{k=1}^{n} z^k \leq 1$. A similar result can also be derived for the DEA with constant returns to scale (i.e., when $\sum_{k=1}^{n} z^k = 1$ is removed from the formulation) when additional constraints are imposed (e.g., maximal bounds on inputs) that will prevent the objective function going to infinity.

Moreover, this result also holds for the Free Disposal Hull (FDH) estimator, proposed by Deprins et al. (1984), because it can be represented as the DEA-VRS problem where $z^k \geq 0$ is replaced with $z^k \in \{0, 1\}$, and so by similar logic we get

$$\widehat{\pi}(w, p|\text{FDH}) = \max\{\pi^1, \ldots, \pi^n\}. \tag{2.3}$$

This result can also be obtained by noting that the maximal profit defined in (1.1) is the same whether it is optimized on a non-convex technology set Ψ or on the 'convexified' version of this technology set and so we have $\widehat{\pi}(w, p|\text{DEA-VRS}) = \widehat{\pi}(w, p|\text{DEA-NIRS}) = \widehat{\pi}(w, p|\text{FDH})$. See Färe and Li (1998) for a related discussion.

It is also worth noting that the latter result, about the equivalence of the (long run) profit maximization with and without the convexity assumption, is different from what is known for other optimizations in economics, e.g., cost minimization, revenue maximization, etc. For example, see Briec et al. (2004) who appear to be the first to prove that cost functions computed with respect to convex technologies are always below those with respect to non-convex technologies. Also, see Balaguer-Coll et al. (2007) and Ang et al. (2018) for further empirical evidence.

3. Concluding remarks

While looking fairly simple (after seeing the proof), this result opens the door for many applications including those where the dimensionality of the output space is too large in comparison with the available sample size, including the so-called 'big wide data' cases.

With this result, the obtained efficiency estimates, in fact, might even have more valuable information about efficiency from an economic point of view, since they incorporate such economically important information as output prices, the corresponding allocation of inputs and the underlying behavior of the decision making units.

For example, for measuring such profit efficiency briefly described above, if one wants to use the DEA framework for estimating Ψ, then the estimated output oriented Farrell-type profit efficiency measure can be obtained as

$$\hat{\mathscr{E}}_o(x^j, y^j; w, p|\widehat{\Psi}) = \frac{c^j}{r^j} + \frac{\max\{\pi^1, \ldots, \pi^n\}}{r^j}, \tag{3.1}$$

where $\max\{\pi^1, \ldots, \pi^n\}$ is the estimated profit function from (2.2), obtained without actual computation of the DEA or FDH model, and it is possible even if the dimensions of inputs and outputs are too large for DEA and FDH to handle, as long as the total revenue and the total costs are observed for the firms of interest. Moreover, note that $\hat{\mathscr{E}}_o(x^j, y^j; w, p|\text{DEA-VRS}) = \hat{\mathscr{E}}_o(x^j, y^j; w, p|\text{DEA-NIRS}) = \hat{\mathscr{E}}_o(x^j, y^j; w, p|\text{FDH})$. In a similar fashion, this result may be also adapted for estimating other profit efficiency measures.

A natural question is what happens if firms face different prices. In this case the result we derived above does not hold in general, yet it may still hold approximately for small variations around the common price benchmark used. This common price benchmark can be viewed as a common reference (e.g., an equilibrium or an average tendency for the prices), in a similar meaning as one selects a common frontier reference for measuring technical efficiency, to make the efficiency scores comparable across different firms. Investigating how much the variation of prices matter theoretically, via simulations or empirically could be a fruitful alley for future research.

Finally, and to conclude this note, among the closely related directions for future research, it is worth noting on the possibility to generalize this result for dynamic cases, e.g., using Bellman equations and involving dynamic productivity indicators (e.g., see Färe and Zelenyuk (2019) and references there in).

Acknowledgment

We thank Knox Lovell, Sung Ko Li and other colleagues for their fruitful feedback. We also thank Australian Research Council (ARC) Future Fellowship grant FT170100401 and The University of Queensland, Australia for their financial support. These individuals and organizations are not responsible for the views expressed in the paper.

References

Ang, F., Mortimer, S.M., Areal, F.J., Tiffin, R., 2018. On the opportunity cost of crop diversification. J. Agric. Econ. 69 (3), 794–814.

Balaguer-Coll, M.T., Prior, D., Tortosa-Ausina, E., 2007. On the determinants of local government performance: A two-stage nonparametric approach. Eur. Econ. Rev. 51 (2), 425–451.

Briec, W., Kerstens, K., Eeckaut, P.V., 2004. Non-convex technologies and cost functions: Definitions, duality and nonparametric tests of convexity. J. Econ. 81 (2), 155–192.

Chambers, R., Chung, Y., Färe, R., 1998. Profit, directional distance functions, and Nerlovian efficiency. J. Optim. Theory Appl. 98 (2), 351–364.

Charnes, A., Cooper, W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European J. Oper. Res. 2 (6), 429–444.

Deprins, D., Simar, L., Tulkens, H., 1984. Measuring labour efficiency in post offices. In: Marchand, M., Pestieau, P., Tulkens, H. (Eds.), The Performance of Public Enterprises: Concepts and Measurement. Springer, Amsterdam, NL, pp. 243–267.

Färe, R., Grosskopf, S., 1985. A nonparametric cost approach to scale efficiency. Scand. J. Econ. 87 (4), 594–604.

Färe, R., He, X., Li, S.K., Zelenyuk, V., 2019. A unifying framework for Farrell efficiency measurement coherent with profit-maximizing principlearrell profit efficiency measurement. Oper. Res. 67 (1), 183–197.

Färe, R., Li, S.K., 1998. Inner and outer approximations of technology: a data envelopment analysis approach. European J. Oper. Res. 105 (3), 622–625.

Färe, R., Zelenyuk, V., 2019. On Luenberger input, output and productivity indicators. Econom. Lett. 179, 72–74.

Farrell, M.J., 1957. The measurement of productive efficiency. J. R. Stat. Soc. Ser. A Gen. 120 (3), 253–290.

Georgescu-Roegen, N., 1951. The aggregate linear production function and its applications to von Neumann's economic model. In: Koopmans, T. (Ed.), Activity Analysis of Production and Allocation. Wiley, New York, NY.

Grifell-Tatjé, E., Lovell, C.A.K., 2015. Productivity Accounting. Cambridge University Press, New York, NY.

Rubinstein, A., 2006. Lecture Notes in Microeconomic Theory. In: Number Gradmicro1 in Online Economics Textbooks, SUNY-Oswego, Department of Economics.

Sickles, R., Zelenyuk, V., 2019. Measurement of Productivity and Efficiency: Theory and Practice. Cambridge University Press, New York, NY.

Simar, L., Wilson, P.W., 2015. Statistical approaches for nonparametric frontier models: A guided tour. Internat. Statist. Rev. 83 (1), 77–110.

Zelenyuk, V., 2020. Aggregation of inputs and outputs prior to Data Envelopment Analysis under big data. European J. Oper. Res. 282 (1), 172–187.